Journal of Statistical Physics, Vol. 98, Nos. 1/2, 2000

Trend to Equilibrium of a Degenerate Relativistic Gas

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Received March 15, 1999; final July 15, 1999

We examine the problem of the trend to equilibrium for a relativistic gas which may follow Fermi–Dirac, Bose–Einsten, classical Boltzmann statistics. We use the relativistic version of the quasiclassical Boltzmann equation for fermions and bosons, the Uehling–Uhlenbeck equation.

KEY WORDS: Kinetic theory; relativity; trend to equilibrium.

1. INTRODUCTION

The study of the trend to equilibrium in kinetic theory is a subject full of pitfalls. The discussion by Truesdell and Muncaster[(1)] on the standard Boltzmann equation, though somehow exaggerated and ungenerous to some authors who have tried to present the topic in a simplified form, gives an idea of the difficulties of the proof.

In order to avoid the pitfalls, we must state the assumptions very clearly and beware of sweeping statements. But this is not enough: even if we do not look for a pedantically rigorous proof, we must be sure that our solution has certain properties, which can only be provided by an accurate existence theory.

The rigorous theory of the Boltzmann equation started in 1933 with a paper by Torsten Carleman, who proved a theorem of global existence and uniqueness for a gas of hard spheres in the so-called space homogeneous case. The theorem was proved under the restrictive assumption that the initial data depend upon the molecular velocity only through its magnitude.[(2)] This restriction is removed in a posthumous book by the same author.[(3)]

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0022-4715/00/0100-0441$18.00/0

The existence theory was extended by D. Morgenstern,[4] who proved a global existence theorem for a gas of Maxwellian molecules in the space homogeneous case, under the assumption that the solution is in L^1 and has finite energy and entropy. His work was further extended by L. Arkeryd in 1972.[5, 6]

The proof of Carleman concerns bounded data (with a weight that ensures a proper decay for large speeds) and proves that the solution remains similarly bounded at later time. His work was later extended by Arkeryd[7] to more general molecular models.

The results of Carleman, Morgenstern and Arkeryd referred to the case in which the solution of the Boltzmann equation does not depend on the space coordinates (the space homogeneous case). Since we shall consider just this case, we shall not mention the remarkable results obtained by many authors in more complicated situations,[8] where the trend to equilibrium may even not hold.

Our aim is to examine the problem when the following generalizations are introduced: a) the gas is relativistic; b) the gas may be degenerate, i.e., follow Fermi–Dirac or Bose–Einstein, rather than Boltzmann statistics. We shall use the relativistic Uehling–Uhlenbeck equation, i.e., the relativistic version of the quasi-classical Boltzmann equation for fermions and bosons.[9, 10]

In the non-relativistic case the Boltzmann equation for a degenerate gas seems to have been studied in a mathematically rigorous way in the case of fermions,[11, 12] but not in the case of bosons. In fact, the Fermi case is easier, in a sense, than the classical Boltzmann case; in fact physics provides, thanks to the exclusion principle, an upper bound to the distribution function. Dolbeault[11] also discusses the trend to equilibrium for a Fermi gas. On the contrary, the well-known phenomenon of Bose condensation indicates that physics, in this case, tends to drive the distribution function toward higher values where these values are already high, thus jeopardizing the argument which ensures the boundedness of the distribution function. A preliminary study has been presented by Lu.[13]

We remark that, in the case of Boltzmann statistics the trend to equilibrium in the space inhomogeneous case has been studied by Andréasson[14] and by Glassey and Strauss.[15]

In this paper we shall discuss the trend to equilibrium in a precise way, under the assumption of boundedness and equicontinuity of the distribution function. This property has been shown for the usual Boltzmann equation in the case of distinguishable particles and fermions, as indicated above. The proofs can be extended (under suitable assumptions on the cross section) to the relativistic case. Actually this extension has never been rigorously studied in detail, but seems to follow from the regularizing properties of the kernel proved by Andréasson.[14] In the case of Bose particles, the property is simply assumed here, the proof being left for future work.

2. THE UEHLING–UHLENBECK EQUATION

We consider a relativistic degenerate ideal gas described by the one-particle distribution function $f(x^\alpha, p^\alpha)$ which is a function of the space-time cooordinates $(x^\alpha)=(ct, \mathbf{x})$ and momentum four-vector $(p^\alpha)=(p^0, \mathbf{p})$ in a Minkowski space characterized by the metric tensor $\eta^{\alpha\beta}$ with signature $(1, -1, -1, -1)$. Due to the constraint of constant lenght of the momentum four-vector $p^\alpha p_\alpha = m^2c^2$ or $p^0=\sqrt{|\mathbf{p}|^2+m^2c^2}$ we have that $f(x^\alpha, p^\alpha)= f(\mathbf{x}, \mathbf{p}, t)$. The element of volume $d^3x\, d^3p$ in the phase space is a scalar invariant and $f(\mathbf{x}, \mathbf{p}, t)\, d^3x\, d^3p$ gives at time t the number of particles in the volume element d^3x about $\mathbf{x}$ and with momenta in the range d^3p about $\mathbf{p}$.

The Uehling–Uhlenbeck equation[9, 10] is a quasi-classical Boltzmann equation that incorporates modifications in the collision term of the Boltzmann equation since the particles obey quantum statistics. For ideal gases in the absence of external forces the relativistic Uehling–Uhlenbeck equation reads:

$$p^\alpha \frac{\partial f}{\partial x^\alpha} = \frac{1}{2}\int \left[f'_* f' \left(1+\varepsilon \frac{f h^3}{g_s}\right)\left(1+\varepsilon \frac{f_* h^3}{g_s}\right) - f_* f \left(1+\varepsilon \frac{f' h^3}{g_s}\right)\left(1+\varepsilon \frac{f'_* h^3}{g_s}\right)\right] F\sigma\, d\Omega\, \frac{d^3 p_*}{p_{*0}} \tag{1}$$

In the above equation σ is an invariant differential cross-section, $d\Omega$ an element of solid angle, $\mathbf{p}$ and $\mathbf{p}_*$ represent the pre-collisional momenta. Further $\mathbf{p}'$ and $\mathbf{p}'_*$ denote pre-collisional momenta which will be transformed into $\mathbf{p}$ and $\mathbf{p}_*$ by a collision. F is the invariant flux

$$F = p_0 p_{*0} \sqrt{\left(\frac{\mathbf{p}}{p_0} - \frac{\mathbf{p}_*}{p_{*0}}\right)^2 - \left(\frac{\mathbf{p}}{p_0} \times \frac{\mathbf{p}_*}{p_{*0}}\right)^2} \tag{2}$$

and f'_* is an abbreviation for $f(\mathbf{x}, \mathbf{p}'_*, t)$ and so on. Further $g_s\, d^3x\, d^3p/h^3$ gives the number of states in the phase space with g_s denoting the degeneracy factor of particles with spin s and rest mass m:

$$g_s = \begin{cases} 2s+1 & \text{for} \quad m \neq 0 \\ 2s & \text{for} \quad m = 0 \end{cases} \tag{3}$$

where h is the Planck constant and ε refers to the statistics:

$$\varepsilon = \begin{cases} +1 & \text{for Bose–Einstein statistics} \\ -1 & \text{for Fermi–Dirac statistics, and} \\ 0 & \text{for Maxwell–Boltzmann statistics} \end{cases} \tag{4}$$

For the relativistic Uehling–Uhlenbeck equation (1) one can obtain a general equation of transfer through the multiplication of (1) by an arbitrary function $\psi(x^\beta, p^\beta)$ and by integrating the resulting equation over all values of d^3p/p_0. After some manipulations we get

$$\begin{aligned} &\frac{\partial}{\partial x^\alpha} \int \psi p^\alpha f \frac{d^3p}{p_0} - \int p^\alpha \frac{\partial \psi}{\partial x^\alpha} f \frac{d^3p}{p_0} \\ &\quad = \frac{1}{8} \int (\psi + \psi_* - \psi' - \psi'_*) \left[f'_* f' \left(1 + \varepsilon \frac{f h^3}{g_s}\right)\left(1 + \varepsilon \frac{f_* h^3}{g_s}\right) \right. \\ &\qquad \left. - f_* f \left(1 + \varepsilon \frac{f' h^3}{g_s}\right)\left(1 + \varepsilon \frac{f'_* h^3}{g_s}\right)\right] F\sigma \, d\Omega \frac{d^3p_*}{p_{*0}} \frac{d^3p}{p_0} \end{aligned} \tag{5}$$

The right-hand side of the above equation follows by the use of the well-known symmetry properties of the collision term.

The macroscopic fields of particle four-flow N^α and energy-momentum tensor $T^{\alpha\beta}$ are defined in terms of the one-particle distribution function through

$$N^\alpha = c \int p^\alpha f \frac{d^3p}{p_0}, \qquad T^{\alpha\beta} = c \int p^\alpha p^\beta f \frac{d^3p}{p_0} \tag{6}$$

The balance equations for these fields are obtained from the general equation of transfer (5) by choosing $\psi = c$ and $\psi = cp^\beta$, respectively

$$\partial_\alpha N^\alpha = 0, \qquad \partial_\beta T^{\alpha\beta} = 0 \tag{7}$$

By choosing in the general equation of transfer (5)

$$\psi = -kc \left[\ln \left(\frac{f h^3}{g_s} \right) - \left(1 + \frac{g_s}{\varepsilon f h^3} \right) \ln \left(1 + \frac{\varepsilon f h^3}{g_s} \right) \right] \tag{8}$$

where k is the Boltzmann constant it follows a balance equation that reads

$$\partial_\alpha S^\alpha = \varsigma \tag{9}$$

where the quantities S^α and ς are defined by

$$S^\alpha = \int p^\alpha f \left[-kc \ln\left(\frac{fh^3}{g_s}\right) + kc\left(1 + \frac{g_s}{\varepsilon f h^3}\right) \ln\left(1 + \frac{\varepsilon f h^3}{g_s}\right)\right] \frac{d^3p}{p_0} \tag{10}$$

$$\begin{aligned}\varsigma = \frac{kc}{8} \int f_* f \left(1 + \varepsilon \frac{f' h^3}{g_s}\right)\left(1 + \varepsilon \frac{f'_* h^3}{g_s}\right) \\ \times \ln \frac{f'_* f'(1+\varepsilon(fh^3/g_s))(1+\varepsilon(f_* h^3/g_s))}{f_* f(1+\varepsilon(f'h^3/g_s))(1+\varepsilon(f'_* h^3/g_s))} \\ \times \left[\frac{f'_* f'(1+\varepsilon(fh^3/g_s))(1+\varepsilon(f_* h^3/g_s))}{f_* f(1+\varepsilon(f'h^3/g_s))(1+\varepsilon(f'_* h^3/g_s))} - 1\right] F\sigma\, d\Omega \frac{d^3p_*}{p_{*0}} \frac{d^3p}{p_0}\end{aligned} \tag{11}$$

Since the distribution functions are non-negative we can use the relation-ships

$$\begin{cases}(1-x)\ln x < 0 & \text{for all} \quad x > 0 \\ (1-x)\ln x = 0 & \text{for} \quad x = 1\end{cases} \tag{12}$$

to infer that, the right-hand side of (9) is non-negative. Hence (9) represents the balance equation of the entropy four-flow S^α and we identify ς as the entropy production which is non-negative. The balance equation for the entropy four-flow is then written as:

$$\partial_\alpha S^\alpha = \varsigma \geqslant 0 \tag{13}$$

It is interesting to write the entropy four-flow for a gas that obeys the Maxwell–Boltzmann statistics. In this case by taking the limit when $\varepsilon \to 0$ in (10), yields

$$S^\alpha = -kc \int p^\alpha f \ln\left(\frac{fh^3}{e g_s}\right) \frac{d^3p}{p_0} \tag{14}$$

3. EQUILIBRIUM

By inspecting the general equation of transfer one concludes that the right-hand side of (5) vanishes if

$$\psi + \psi_* = \psi' + \psi'_* \tag{15}$$

A function that satisfies (15) is called a summational invariant. One can prove (see, for example, ref. 16) that a summational invariant is a linear combination of the momentum four-vector, e.g.,

$$\psi = A + B^{\alpha} p_{\alpha} \tag{16}$$

where A is a scalar and B^{α} a four-vector that do not depend on p^{α}. One can easily verify that (16) is a sufficient condition for (15), since the conservation of the momentum four-vectors $p^{\alpha} + p_{*}^{\alpha} = p'^{\alpha} + p'^{\alpha}_{*}$ must hold.

From the above result we are ready to derive the expression for the equilibrium distribution function. Indeed, in equilibrium the entropy source must vanish and hence the integrand in the right-hand side of (1) must vanish, the number of particles entering the volume element $d^3x\, d^3p$ in the phase space being equal to the number of particles that leave it. Hence we have that

$$\begin{aligned} f'^{(0)}_{*} f'^{(0)} & \left(1 + \varepsilon \frac{f^{(0)} h^3}{g_s}\right)\left(1 + \varepsilon \frac{f^{(0)}_{*} h^3}{g_s}\right) \\ &= f^{(0)}_{*} f^{(0)} \left(1 + \varepsilon \frac{f'^{(0)} h^3}{g_s}\right)\left(1 + \varepsilon \frac{f'^{(0)}_{*} h^3}{g_s}\right) \end{aligned} \tag{17}$$

where the index (0) denotes the equilibrium value of the one-particle distribution function. By taking the logarithm of the above expression it follows that

$$\begin{aligned} \ln & \left(\frac{f^{(0)}}{1 + \varepsilon (f^{(0)} h^3 / g_s)}\right) + \ln \left(\frac{f^{(0)}_{*}}{1 + \varepsilon (f^{(0)}_{*} h^3 / g_s)}\right) \\ &= \ln \left(\frac{f'^{(0)}_{E}}{1 + \varepsilon (f'^{(0)} h^3 / g_s)}\right) + \ln \left(\frac{f'^{(0)}_{*}}{1 + \varepsilon (f'^{(0)}_{*} h^3 / g_s)}\right) \end{aligned} \tag{18}$$

Hence $\ln[f^{(0)}/(1 + \varepsilon f^{(0)} h^3/g_s)]$ is a summational invariant and according to (15) it must be a linear combination of the momentum four-vector p^{α}:

$$\ln \left(\frac{f^{(0)}}{1 + \varepsilon (f^{(0)} h^3 / g_s)}\right) = -(A + B^{\alpha} p_{\alpha}), \qquad \text{or} \qquad f^{(0)} = \frac{g_s/h^3}{e^{-a + B^{\alpha} p_{\alpha}} - \varepsilon} \tag{19}$$

where $a = -A - \ln(g_s/h^3)$.

We shall now determine the two unknows a and B^{α} of the equilibrium distribution function $(19)_2$. We start by introducing a frame, to be called the Lorentz rest frame and denoted by an index R, in which the gas is seen as an isotropic body for an observer that moves with the gas velocity $\mathbf{v}$. In this frame the four-velocity of the gas

$$(U^\alpha)=\left(\frac{c}{\sqrt{1-v^2/c^2}},\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}}\right),\qquad \text{such that}\quad U^\alpha U_\alpha=c^2 \tag{20}$$

reduces to

$$(U_R^\alpha)=(c,0,0,0)=(c,\mathbf{0}) \tag{21}$$

In the Lorentz rest frame we introduce the quantities:

$$\begin{cases} n\text{—particle number density}\\ p\text{—isotropic pressure}\\ e\text{—internal energy per particle}\\ T\text{—temperature}\\ s\text{—entropy per particle}\end{cases} \tag{22}$$

The particle number density is related to the number of baryons minus the number of antibaryons per unit three-dimensional proper volume.

In terms of the four-velocity of the gas U^α the equilibrium values of the particle four-flow, energy-momentum tensor, and entropy four-flow in an arbitrary Lorentz frame read

$$N_E^\alpha=nU^\alpha,\qquad T_E^{\alpha\beta}=-p\eta^{\alpha\beta}+(en+p)\frac{U^\alpha U^\beta}{c^2},\qquad S_E^\alpha=ns_E U^\alpha \tag{23}$$

where the index E denotes the equilibrium value of the quantity.

We assume also that in the Lorentz rest frame B^α has only the time component. Thus we write

$$(B_R^\alpha)=\left(\frac{\zeta}{mc},\mathbf{0}\right) \tag{24}$$

where ζ is a parameter which we shall identify latter. Since B^α is a four-vector we have that

$$B_R^\alpha B_{R\alpha}=B^\alpha B_\alpha=\frac{\zeta^2}{(mc)^2},\qquad \text{and}\qquad \frac{\partial\zeta}{\partial B_\alpha}=\frac{(mc)^2}{\zeta}B^\alpha \tag{25}$$

By inserting the equilibrium distribution function $(19)_2$ into the definition of the particle four-flow $(6)_1$ it follows that

$$N_E^\alpha = c \int p^\alpha \frac{g_s/h^3}{e^{-a + B^\alpha p_\alpha} - \varepsilon} \frac{d^3 p}{p_0} \tag{26}$$

Let $\mathscr{I}$ be the following integral

$$\mathscr{I} = \int \frac{g_s/h^3}{e^{-a + B^\alpha p_\alpha} - \varepsilon} \frac{d^3 p}{p_0} \tag{27}$$

If we differentiate N_E^α with respect to a and $\mathscr{I}$ with respect to B_α we get that

$$\frac{\partial N_E^\alpha}{\partial a} = -c \frac{\partial \mathscr{I}}{\partial B_\alpha} \tag{28}$$

On the other hand, in a Lorentz rest frame we can write

$$n = 4\pi (mc)^3 \frac{g_s}{h^3} J_{21}, \qquad \mathscr{I} = 4\pi (mc)^2 \frac{g_s}{h^3} J_{20} \tag{29}$$

where J_{nm} denotes the integral

$$J_{nm}(\zeta, a) = \int_0^\infty \frac{\sinh^n x \cosh^m x}{e^{\zeta \cosh x - a} - \varepsilon} dx \tag{30}$$

By differentiating n with respect to a and $\mathscr{I}$ with respect to ζ and by using the following relationships

$$\begin{aligned} \frac{\partial J_{nm}}{\partial a} &= \frac{n-1}{\zeta} J_{n-2, m+1} + \frac{m}{\zeta} J_{n, m-1} \\ \frac{\partial J_{nm}}{\partial \zeta} &= -\frac{n-1}{\zeta} J_{n-2, m+2} - \frac{m+1}{\zeta} J_{n, m} \end{aligned} \tag{31}$$

it follows that

$$\frac{\partial n}{\partial a} = -\frac{1}{mc} \frac{\partial \mathscr{I}}{\partial \zeta} \tag{32}$$

Now we get by combining equations (23), $(25)_2$, (28) and (32) that B^α is given by

$$B^\alpha = \frac{\zeta}{mc^2} U^\alpha \tag{33}$$

Hence we have identified B^α, and we proceed to identifying ζ and a. If we insert the equilibrium distribution function $(19)_2$ into the definition of the energy-momentum tensor $(6)_2$, and consider a Lorentz rest frame, we get

$$ne = 4\pi m^4 c^5 \frac{g_s}{h^3} J_{22}, \qquad p = \frac{4\pi}{3} m^4 c^5 \frac{g_s}{h^3} (J_{22} - J_{20}) = \frac{4\pi}{3} m^4 c^5 \frac{g_s}{h^3} J_{40} \tag{34}$$

by using of the relationships

$$ne = T_E^{\alpha\beta} U_\alpha U_\beta, \qquad -3p + ne = T_E^{\alpha\beta} \eta_{\alpha\beta} \tag{35}$$

Since it is not possible to obtain from (34) explicit expressions for a and ζ, we calculate the the equilibrium value of the entropy per particle s_E which follows from $(23)_3$

$$s_E = \frac{1}{nc^2} S_E^\alpha U_\alpha \tag{36}$$

Hence we obtain from (10), $(19)_2$ and (36):

$$s_E = k \left(\frac{\zeta}{mc^2} e - a + \frac{4\pi\zeta}{3} \frac{m^3 c^3}{n} \frac{g_s}{h^3} J_{40} \right) \tag{37}$$

Now the differential of the above equation leads to

$$ds_E = \frac{k\zeta}{mc^2} \left(de - \frac{p}{n^2} dn \right) \tag{38}$$

by using the relationships (31) and (34). We compare (38) with the Gibbs equation

$$ds_E = \frac{1}{T} \left(de - \frac{p}{n^2} dn \right) \tag{39}$$

and identify, $\zeta = mc^2/kT$ as the ratio between the rest energy mc^2 of a particle and kT, which gives the order of magnitude of the thermal energy of the gas.

Further (37) can be written as

$$a = \frac{1}{kT}\left(e - Ts_E + \frac{p}{n}\right) = \frac{\mu_E}{kT} \tag{40}$$

that is a is identified as the ratio between the chemical potential in equilibrium $\mu_E = e - Ts_E + p/n$ and the thermal energy of the gas kT.

Hence we have identified a, ζ and B^α and the equilibrium distribution function $(19)_2$ can be written as:

- relativistic Maxwell–Boltzmann statistics:

$$f^{(0)} = \frac{g_s}{h^3}\, e^{(\mu_E/kT) - (U^\alpha p_\alpha/kT)} \tag{41}$$

- relativistic Fermi–Dirac $(+)$ and Bose–Einstein $(-)$ statistics:

$$f^{(0)} = \frac{g_s/h^3}{e^{-(\mu_E/kT) + (U^\alpha p_\alpha/kT)} \pm 1} \tag{42}$$

The expression (41) was obtained by Jüttner[17] in 1911 and the expression (42) was also obtained by him[18] in 1928.

We remark that the relation between the temperature and the thermal energy is not linear and thus one cannot speak of a "temperature" for a nonequilibrium gas, as one frequently does in the nonrelativistic case.

4. TREND TO EQUILIBRIUM

Before we proceed the analysis of the trend to equilibrium of a relativistic gas we shall introduce some inequalities that will be used in this section. Let y be a positive real variable; then the following inequality holds:

$$(y-1) - \ln y \geqslant 0, \qquad \text{or with} \quad y = \frac{1}{x}, \qquad x \ln x + 1 - x \geqslant 0 \tag{43}$$

Further one can always find a constant $\mathscr{C}$ such that the following inequality holds

$$x \ln x + 1 - x - \mathscr{C}\mathscr{G}(|x-1|)\,|x-1| \geqslant 0 \tag{44}$$

where

$$\mathscr{G}(|x-1|) = \begin{cases} |x-1|, & \text{if} \quad 0 \leqslant |x-1| \leqslant 1 \\ 1 & \text{if} \quad |x-1| \geqslant 1 \end{cases} \tag{45}$$

We return to the balance equation for the entropy four-flow (13) and write it as

$$\partial_\alpha \mathscr{H}^\alpha = \mathscr{S} \leqslant 0, \qquad \text{with} \quad \mathscr{H}^\alpha = -S^\alpha/k, \qquad \mathscr{S} = -\varsigma/k \tag{46}$$

By considering that the one-particle distribution function does not depend on the space coordinates and introducing the scalar invariant

$$\mathscr{H} = \frac{1}{c^2} U_R^\alpha \mathscr{H}_\alpha = \frac{1}{c} \mathscr{H}^0$$
$$= \int p^0 f \left[\ln\left(\frac{fh^3}{g_s}\right) - \left(1 + \frac{g_s}{\varepsilon f h^3}\right) \ln\left(1 + \frac{\varepsilon f h^3}{g_s}\right)\right] \frac{d^3p}{p_0} \tag{47}$$

we get that the inequality (46) reduces to

$$\frac{d\mathscr{H}}{dt} = \mathscr{S} \leqslant 0 \tag{48}$$

From the above equation one infers that $\mathscr{H}$ decreases. Further the time derivative of $\mathscr{H}$ vanishes when the one-particle distribution function is the equilibrium distribution function, which will be denoted by $\mathscr{H}_E \equiv \mathscr{H}(f^{(0)})$.

Now we choose in the inequalities (43)

$$y = \frac{1 + \varepsilon f h^3/g_s}{1 + \varepsilon f^{(0)} h^3/g_s}, \qquad \text{and} \qquad x = \frac{fh^3/g_s}{1 + \varepsilon f h^3/g_s} \frac{1 + \varepsilon f^{(0)} h^3/g_s}{f^{(0)} h^3/g_s} \tag{49}$$

and get by adding the two resulting inequalities:

$$f\left[\ln \frac{fh^3/g_s}{1 + \varepsilon f h^3/g_s} - \ln \frac{f^{(0)} h^3/g_s}{1 + \varepsilon f^{(0)} h^3/g_s}\right] - \frac{g_s}{\varepsilon h^3} \ln\left(\frac{1 + \varepsilon f h^3/g_s}{1 + \varepsilon f^{(0)} h^3/g_s}\right) \geqslant 0 \tag{50}$$

We multiply (50) by p^0 and integrate the resulting equation over all values of d^3p/p_0 and get

$$\mathscr{H} - \mathscr{H}_E \geqslant \int p^0 (f - f^{(0)}) \ln \frac{f^{(0)} h^3/g_s}{1 + \varepsilon f^{(0)} h^3/g_s} \frac{d^3p}{p_0}$$
$$= -\frac{U_R^\alpha}{c} \int p_\alpha (f - f^{(0)}) [A + B^\beta p_\beta] \frac{d^3p}{p_0} \tag{51}$$

by the use of $(19)_1$. The right-hand side of the above inequality vanishes if we impose that

$$U_\alpha N^\alpha = U_\alpha N_E^\alpha, \qquad U_\alpha U_\beta T^{\alpha\beta} = U_\alpha U_\beta T_E^{\alpha\beta} \tag{52}$$

The conditions (52) imply that

$$\mathscr{H} \geqslant \mathscr{H}_E \tag{53}$$

We base on Carleman[2, 3] and Cercignani[19] to prove the following theorem.

Theorem 1. If $\mathscr{H}(t)$ is a continuous and differentiable function of t that satisfies (48) and (53) and f is uniformly bounded and equicontinuous in p_α, and $\int p_0^{1+\eta} f\, d^3p$ $(\eta > 0)$ is uniformly bounded, then $\mathscr{H}$ tends to $\mathscr{H}_E$ when $t \to \infty$.

Proof. From (53) and (48) we infer that $\mathscr{H}$ is bounded from below by $\mathscr{H}_E$, its derivative being negative and vanishing when the one-particle distribution function is the equilibrium one. Hence it is possible to find a sequence of instants of time $t_1, t_2, \ldots, t_n, \ldots$ such that

$$\lim_{n\to\infty} \frac{d\mathscr{H}}{dt}(t_n) = 0 \tag{54}$$

Further, because of Ascoli–Arzelà's theorem, there exists a uniformly converging sequence $f(t_n) \equiv f_n$ such that on any compact set $\mathscr{D}$

$$\lim_{n\to\infty} f_n = f_\infty \tag{55}$$

If we prove that

$$f'_{*\infty} f'_\infty \left(1 + \varepsilon \frac{f_\infty h^3}{g_s}\right)\left(1 + \varepsilon \frac{f_{*\infty} h^3}{g_s}\right) = f_{*\infty} f_\infty \left(1 + \varepsilon \frac{f'_\infty h^3}{g_s}\right)\left(1 + \varepsilon \frac{f'_{*\infty} h^3}{g_s}\right) \tag{56}$$

then according to (17) f_∞ is an equilibrium distribution function and, as we shall prove, $\mathscr{H}$ tends to $\mathscr{H}_E$ when $t \to \infty$.

In order to prove (56) we suppose that there exists a domain of positive measure $\mathscr{D}$ such that

$$\left| f'_{*\infty} f'_\infty \left(1 + \varepsilon \frac{f_\infty h^3}{g_s}\right)\left(1 + \varepsilon \frac{f_{*\infty} h^3}{g_s}\right) - f_{*\infty} f_\infty \left(1 + \varepsilon \frac{f'_\infty h^3}{g_s}\right)\left(1 + \varepsilon \frac{f'_{*\infty} h^3}{g_s}\right) \right| \geqslant R > 0 \tag{57}$$

where R is a constant. The uniform convergence of f_n to f_∞ implies that it is possible to find a n_0 such that

$$\left| f'_{*n} f'_n \left(1+\varepsilon \frac{f_n h^3}{g_s}\right)\left(1+\varepsilon \frac{f_{*n} h^3}{g_s}\right) \right. \\ \left. - f_{*n} f_n \left(1+\varepsilon \frac{f'_n h^3}{g_s}\right)\left(1+\varepsilon \frac{f'_{*n} h^3}{g_s}\right) \right| > \frac{R}{2} > 0 \tag{58}$$

for $n > n_0$ in $\mathscr{D}$. Hence it follows that

$$\left| f'_{*n} f'_n \left(1+\varepsilon \frac{f_n h^3}{g_s}\right)\left(1+\varepsilon \frac{f_{*n} h^3}{g_s}\right) - f_{*n} f_n \left(1+\varepsilon \frac{f'_n h^3}{g_s}\right)\left(1+\varepsilon \frac{f'_{*n} h^3}{g_s}\right) \right| \\ \times \left| \ln \frac{f'_{*n} f'_n (1+\varepsilon(f_n h^3/g_s))(1+\varepsilon(f_{*n} h^3/g_s))}{f_{*n} f_n (1+\varepsilon(f'_n h^3/g_s))(1+\varepsilon(f'_{*n} h^3/g_s))} \right| \\ > \frac{R}{2} \ln\left(1+\frac{R}{2M^2(1+\varepsilon M h^3/g_s)^2}\right) \tag{59}$$

by considering that the one-particle distribution function f is bounded by a constant M.

If we multiply (59) by $(c/8)\, F\sigma\, d\Omega (d^3p_*/p_{*0})(d^3p/p_0)$ and integrate the resulting equation over all values of d^3p_*/p_{*0} and d^3p/p_0 we get

$$-\frac{d\mathscr{H}}{dt}(t_n) > \frac{Rc}{16} \ln\left(1+\frac{R}{2M^2(1+\varepsilon M h^3/g_s)^2}\right) \int_{\mathscr{D}} F\sigma\, d\Omega \frac{d^3p_*}{p_{*0}} \frac{d^3p}{p_0} \tag{60}$$

by the use of (11), (46)–(48). If we take the limit of the above expression when $n \to \infty$, yields

$$0 < -\frac{Rc}{16} \ln\left(1+\frac{R}{2M^2(1+\varepsilon M h^3/g_s)^2}\right) \int_{\mathscr{D}} F\sigma\, d\Omega \frac{d^3p_*}{p_{*0}} \frac{d^3p}{p_0} \tag{61}$$

which contradicts the premise that $\mathscr{D}$ has a non-zero measure. Hence f_∞ is a Maxwellian; in order to prove the theorem we need to show that f_∞ is the Maxwellian f_E determined by the initial data and the conservation laws; this is easy because we can pass to the limit under the integral sign in the expression of the conserved moments thanks to the assumption that $\int p_0^{1+\eta} f\, d^3p$ $(\eta > 0)$ is uniformly bounded. Then $\mathscr{H}$ tends to $\mathscr{H}_E$ and f tends to f_E when $t \to \infty$ along an arbitrary sequence. This is based on the well-known fact that under the constraints provided by the conservation laws, $\mathscr{H}$ is a convex functional with $\mathscr{H}_E$ as its minimum (attained for $f = f_E$). The theorem is proved.

We have proved that f tends to f_E in the middle of the previous proof. It is, however, interesting to prove (see the comments after the proof):

Theorem 2. If $\mathscr{H}$ tends to $\mathscr{H}_E$ when $t \to \infty$, then f tends strongly to $f^{(0)}$ in L^1.

Proof. We begin by writing instead of (50) the inequality

$$f\left[\ln \frac{fh^3/g_s}{1+\varepsilon fh^3/g_s} - \ln \frac{f^{(0)}h^3/g_s}{1+\varepsilon f^{(0)}h^3/g_s}\right] - \frac{g_s}{\varepsilon h^3} \ln\left(\frac{1+\varepsilon fh^3/g_s}{1+\varepsilon f^{(0)}h^3/g_s}\right)$$
$$-\mathscr{C}\mathscr{G}\left(\frac{|f^{(0)}-f|}{f^{(0)}(1+\varepsilon fh^3/g_s)}\right)\frac{|f^{(0)}-f|}{1+\varepsilon f^{(0)}h^3/g_s} \geqslant 0 \tag{62}$$

which is a consequence of (44). Following the same procedure as above in order to derive (51) we get

$$\mathscr{H} - \mathscr{H}_E \geqslant \mathscr{C}\left[\int_{L_t} p^0 \frac{|f^{(0)}-f|}{1+\varepsilon f^{(0)}h^3/g_s} \frac{d^3p}{p_0}\right.$$
$$\left.+\int_{S_t} p^0 \frac{|f^{(0)}-f|^2}{f^{(0)}(1+\varepsilon fh^3/g_s)(1+\varepsilon f^{(0)}h^3/g_s)} \frac{d^3p}{p_0}\right] \tag{63}$$

In the above inequality L_t and S_t denote the integration domains where $|f^{(0)}-f|$ is larger or smaller than $f^{(0)}$, respectively. Since we have assumed that $\mathscr{H}$ tends to $\mathscr{H}_E$ when $t \to \infty$ and the integrands are positive, both integrals on the right-hand abide of (63) must tend to zero in this limit. Further by using the Schwarz's inequality one can show that:

$$\int_{S_t} p^0 \frac{|f^{(0)}-f|}{1+\varepsilon f^{(0)}h^3/g_s} \frac{d^3p}{p_0}$$
$$\leqslant \left[\int_{S_t} p^0 \frac{f^{(0)}(1+\varepsilon fh^3/g_s)}{(1+\varepsilon f^{(0)}h^3/g_s)} \frac{d^3p}{p_0}\right]^{1/2}$$
$$\times\left[\int_{S_t} p^0 \frac{|f^{(0)}-f|^2}{f^{(0)}(1+\varepsilon fh^3/g_s)(1+\varepsilon f^{(0)}h^3/g_s)} \frac{d^3p}{p_0}\right]^{1/2} \to 0 \tag{64}$$

when $t \to \infty$. Hence the following integral over all domain of integration tends to zero when $t \to \infty$,

$$\int p^0 \frac{|f^{(0)}-f|}{1+\varepsilon f^{(0)}h^3/g_s}$$
$$= \int_{L_t} p^0 \frac{|f^{(0)}-f|}{1+\varepsilon f^{(0)}h^3/g_s} \frac{d^3p}{p_0} + \int_{S_t} p^0 \frac{|f^{(0)}-f|}{1+\varepsilon f^{(0)}h^3/g_s} \frac{d^3p}{p_0} \to 0 \tag{65}$$

From the above equation we conclude that f tends strongly to $f^{(0)}$ in L^1, proving the above theorem.

This proof of this theorem is less technical than the previous one and hence more appealing to a physicist than that of Theorem 1. There is the strong assumption, however, that $\mathscr{H}$ tends to $\mathscr{H}_E$. A simple proof of this result is needed; to this aim, one of the authors conjectured (in the classical case) an inequality on the entropy source[20] which would lead to an exponential decay of the entropy to its equilibrium value. This inequality has been disproved by several counterexamples if only mass, energy and entropy are assumed to exist at $t=0$;[21, 24] an example where the entropy dissipation rate is arbitrarily low was recently supplied by Bobylev and Cercignani.[25] A modified form of the inequality, which still serves the purpose, has been proved by Toscani and Villani.[26] Entropic convergence to equilibria for general initial data was, however, first discussed by Carlen and Carvalho[22] and inequalities showing that entropy converges in broad generality were established later by the same authors.[23]

5. CONCLUDING REMARKS

Under the impetus of astrophysical research, the relativistic kinetic theory has been the object of a renewed interest in the last few years. One of the simplest problems in kinetic theory is the trend to equilibrium. Here we examined this problem for a relativistic gas, which may follow Fermi–Dirac or Bose–Einsten, or the classical Boltzmann statistics, using the relativistic version of the Uehling–Uhlenbeck equation.

REFERENCES

1. C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas* (Academic Press, New York, 1980).
2. T. Carleman, Sur la théorie de l'équation intégro-differentielle de Boltzmann, *Acta Mathematica* **60**:91–146 (1933).
3. T. Carleman, *Problèmes mathématiques dans la théorie cinétique des gaz* (Almqvist & Wiksells, Uppsala, 1957).
4. D. Morgenstern, General existence and uniqueness proof for spatially homogeneous solutions of the Maxwell-Boltzmann equation in the case of Maxwellian molecules, *Proceedings of the National Academy of Sciences (U.S.A.)* **40**:719–721 (1954).
5. L. Arkeryd, On the Boltzmann equation. Part I: Existence, *Arch. Rat. Mech. Anal.* **45**:1–16 (1972).
6. L. Arkeryd, On the Boltzmann equation. Part II: The full initial value problem, *Arch. Rat. Mech. Anal.* **45**:17–34 (1972).
7. L. Arkeryd, L^∞ estimates for the space-homogeneous Boltzmann equation, *J. Stat. Phys.* **31**:347–361 (1983).

8. C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases* (Springer, New York, 1994).
9. E. A. Uehling, and G. E. Uhlenbeck, Transport phenomena in Einstein–Bose and Fermi–Dirac gases. I, *Phys. Rev.* **43**:552–561 (1933).
10. S. Chapman, and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge Univ. Press, London, 1940).
11. J. Dolbeault, Kinetic models and quantum effects: A modified Boltzmann equation for Fermi–Dirac particles, *Arch. Ration. Mech. Anal.* **127**:101–131 (1994).
12. P. L. Lions, Compactness in Boltzmann's equation via Fourier integral operator and applications III, *J. Math. Kyoto Univ.* **34**:539–584 (1994).
13. X. Lu, Global Isotropic Solutions of a Modified Boltzmann Equation for Bose–Einstein Particles, Presented at the XVI International Conference on Transport Theory (Atlanta, May 10–15, 1999).
14. H. Andréasson, Regularity of the gain term and strong L^1 convergence to equilibrium for the relativistic Boltzmann equation, *SIAM Math. Anal.* **27**:1387–1405 (1996).
15. R. Glassey and W. Strauss, Asymptotic stability of the relativistic equilibrium, *Publ. Res. Inst. Math. Sci.* **29**:301–347 (1992).
16. C. Cercignani and G. M. Kremer, On relativistic collisional invariants, *J. Stat. Phys.* (to appear, 1999).
17. F. Jüttner, Das Maxwellsche gesetz der geschwindigkeitsverteilung in der relativtheorie, *Ann. Physik Chemie* **34**:856–882 (1911).
18. F. Jüttner, Die relativistische quantentheorie des idealen gases, *Zeitschr. Physik* **47**:542–566 (1928).
19. C. Cercignani, *Ludwig Boltzmann. The Man Who Trusted Atoms* (Oxford University Press, Oxford, 1998).
20. C. Cercignani, H-theorem and trend to equilibrium in the kinetic theory of gases, *Arch. Mech.* **34**:231–241 (1982).
21. A. V. Bobylev, Exact solutions of the Boltzmann equation and the theory of relaxation of Maxwell gas, *USSR Theor. Math. Phys.* **60**:280–310 (1984).
22. E. A. Carlen and M. C. Carvalho, The central limit theorem, entropy production and stability of the rate of convergence for the Boltzmann equation, in *Probabilistic Methods in Mathematical Physics*, F. Guerra, M. I. Loffredo, and C. Marchioro, eds. (World Scientific, Singapore, 1992), pp. 140–154.
23. E. A. Marten and M. C. Carvalho, Entropy production estimates for Boltzmann equations with physically realistic collision kernels, *J. Stat. Phys.* **74**:743–782 (1994).
24. B. Wennberg, Entropy dissipation and moment production for the Boltzmann equation, *J. Stat. Phys.* **86**:1053–1066 (1997).
25. A. V. Bobylev and C. Cercignani, On the rate of entropy production for the Boltzmann equation, *J. Stat. Phys.* (to appear, 1999).
26. G. Toscani and C. Villani, Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation, submitted to *Comm. Math. Phys.* (1998).